Aerospace Combustion

Lecture 8:

Fluid Mechanics: Basics and Turbulence

Content

- Basics of Transport and Governing Equations
  - Reynolds and Favre-averaged Navier-Stokes Equations
- Turbulence, what is it ?
  - Energy Spectrum
  - Kolomogorov Hypothesis of Homogenous Isotropic Turbulence
  - Reynolds Averaged NS
- Turbulence Models
  - Eddy viscosity models
  - Reynolds-Stress models

Fluid Mechanic Basics

Basic Transport Phenomena

In general, any gradient of pressure, velocity, temperature or concentration initiates a transport of mass, momentum, or energy

Flux \ Driving Force	Pressure Gradient	Velocity Gradient	Temperature Gradient	Concentration Gradient
Momentum		Newton [μ]		
Energy			Fourier [λ]	Dufour [D_i^c]
Mass	Bernoulli [ρ]		Sorret [D_i^T]	Fick [D]

$$\tau = \mu \frac{\partial u}{\partial x}$$

$$\dot{q} = -\lambda A \frac{dT}{dx}$$

$$\Delta p = \frac{1}{2}\rho\left(u_1^2 - u_2^2\right)$$

$$\frac{\partial \omega}{\partial x_i} = -\frac{D_T}{D}\omega_0\left(1-\omega_0\right)\frac{\partial T}{\partial x_i}$$

$$j = -D\frac{dc}{dx}$$

with ω the mass fraction,

ω_0 the initial mass fraction

Typically, the Dufour effect can be neglected. However, the Sorret effect can be important for some molecules in particular H and H2 at low to moderate temperatures.

General description of Soret effect in flames:

- Lighter molecules will move to high temperature region
- heavier ones to low temperature region.

Fluid Mechanic Basics

Basic Equations

Conservation of mass and momentum for a fluid in form under the assumption that the fluid is a continuum (no need to treat individual particles or molecules and all fields , i.e. flow velocity, pressure, temperature, density are differentiable)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0;$$

Continuity equation

$$\rho \frac{Du}{Dt} = \nabla \sigma + f;$$

Cauchy momentum equation with σ Cauchy stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_{ii} & \tau_{ij} & \tau_{ik} \\ \tau_{jk} & \sigma_{jj} & \tau_{jk} \\ \tau_{ki} & \tau_{kj} & \sigma_{kk} \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{ii} + \mathrm{p} & \tau_{ij} & \tau_{ik} \\ \tau_{jk} & \sigma_{jj} + \mathrm{p} & \tau_{jk} \\ \tau_{ki} & \tau_{kj} & \sigma_{kk} + p \end{pmatrix};$$

$$\rho \frac{Du}{Dt} = \nabla p + \nabla \tau + f;$$

with σ normal stresses and τ shear stresses

Fluid Mechanic Basics

Basic Equations

$$\rho \frac{Du}{Dt} = \nabla p + \nabla \tau + f;$$

Newton made the observation that in many fluids $\tau \propto \frac{\partial u}{\partial y}$

Assumptions:

- Stress tensor is linear function of strain rate tensor or velocity gradient
- Fluid is isotropic
- For a fluid at rest $\nabla \tau = 0$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \frac{\partial u_k}{\partial x_k}$$ Newtonian fluid

with μ shear viscosity and λ bulk (volume) viscosity* which describes a viscous effect associated with a volume change

Quite often the λ term with is neglected but it still isn't clear if this is correct for all application.

If not taken as zero, it usually is taken as $\lambda = -\frac{2}{3}\mu$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} \right) + \rho g_j;$$

Navier-Stokes equations for a compressible Newtonian fluid

Basic Equations

Navier-Stokes equations for an incompressible Newtonian fluid

Assumptions:

- Shear viscosity μ = const., second viscosity effect $\lambda = 0$

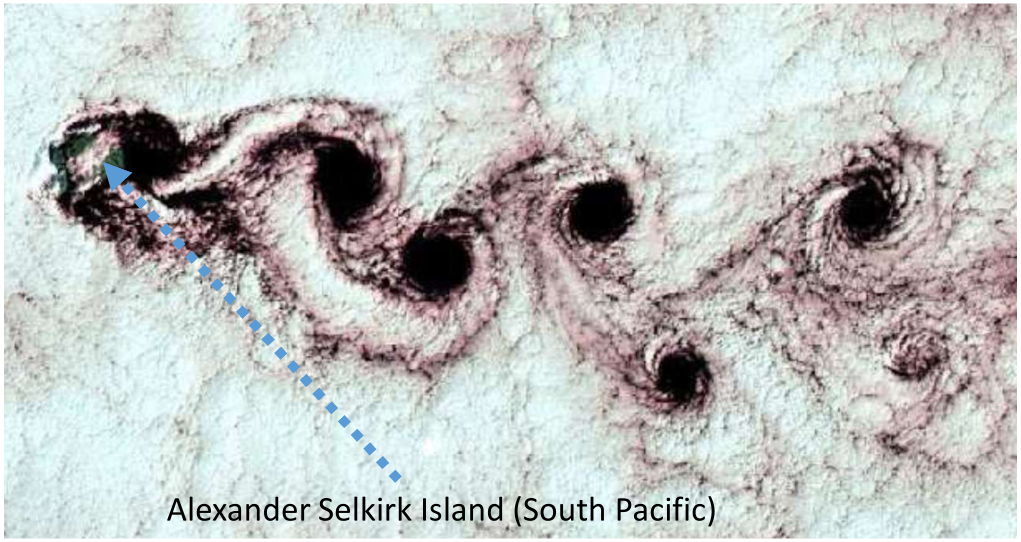
$$\rho \frac{\partial}{\partial t}\left(u_j\right) + \rho \frac{\partial}{\partial x_i}\left(u_i u_j\right) = -\frac{\partial p}{\partial x_j} + \mu\left(\frac{\partial^2 u_i}{\partial x_j^2}\right) + \rho g_j;$$

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i u_j\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_i}{\partial x_j^2}\right) + g_j;$$

Fluid Mechanic Basics

Turbulence: What are its characteristics ?

- .. is a feature of the fluid flow and not the fluid.
- .. is a continuum phenomenon (smallest eddies are much larger than molecular scales).
- .. is irregular and chaotic and occurs at high Reynolds numbers
- Turbulent flows are rotational, they have non-zero vorticity.
- Diffusivity of turbulence is reason for increased momentum, mass and heat transfer.
- Turbulent flows are always chaotic but not all chaotic flows are turbulent.
- They are highly diffusive. If a flow looks random but doesn't show spreading of velocity fluctuation, then it's not turbulent.
- They are highly dissipative and die out quickly without energy addition.



Kolmogorov Hypothesis

1. Local isotropy: At sufficiently high Reynolds numbers, small – scale turbulent motions are statistically isotropic
2. Similarity 1: In every turbulent flow at sufficient high Re numbers, the statistics of the small – scale motions have a universal form which is uniquely determined by the viscosity ν and the dissipation rate (ε).
3. Similarity 2: In every turbulent flow at sufficient high Re numbers, the statistics of the motions in the inertial range have a universal form which is uniquely determined by (ε), independent of ν.

Large (integral) scale:
Statistics aren't universal, depend on large-scale forcing

Inertial range:
Statistics universal, depend on (ε)

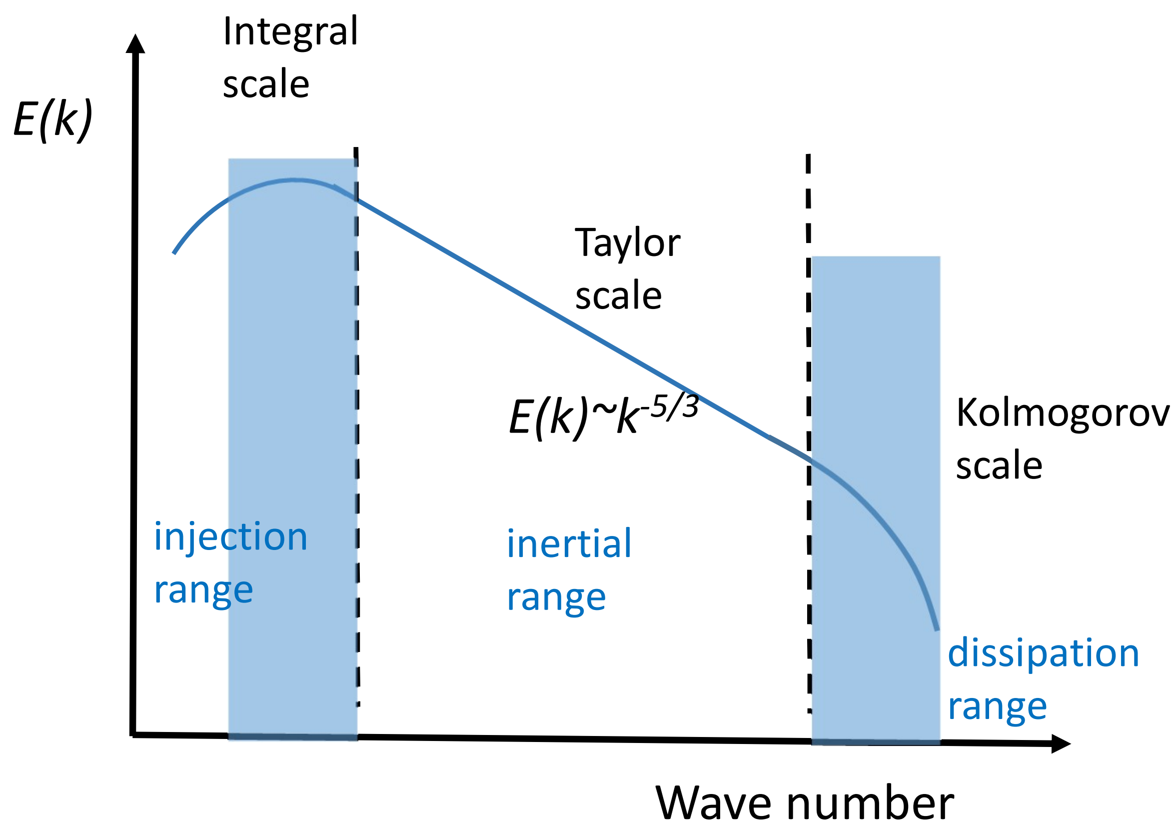
Dissipative range:
Statistics universal, depend on (ε) and ν

Fluid Mechanic Basics

Turbulence Energy Cascade

- Energy is transported from large eddies to smaller ones
- Length scales
  - Largest eddies ($k^{3/2}/\varepsilon$)
  - Isotropic turbulence: Taylor micro-scale $(15\,\nu u'^2/\varepsilon)^{1/2}$
  - smallest eddies: Kolmogorov length scale $(\nu^3/\varepsilon)^{1/4}$

The Kolmogorov length scale is usually defined as the product of a velocity scale $(\nu\,\varepsilon)^{1/4}$ and a time scale $(\nu/\varepsilon)^{1/2}$



Fluid Mechanic Basics

Turbulence

Homogenous turbulence means:

All statistic flow characteristics are equal
→ transitional invariance

$$\overline{u_i'^2} = C_1,\ \overline{u_j'^2} = C_2,\ \overline{u_k'^2} = C_3$$

Isotropic turbulence means:

All statistic flow characteristics are directionally independent
→ rotational invariance

$$\overline{u_i'^2} = \overline{u_j'^2} = \overline{u_k'^2}$$

$$\tau_{ii} = \rho\overline{u_i'^2} = \rho\overline{u_j'^2} = \rho\overline{u_k'^2} = const$$

$$\tau_{ij} = \rho\overline{u_i'u_j'} = \rho\overline{u_i'u_k'} = \rho\overline{u_j'u_k'} = 0$$

Homogenous isotropic turbulence means:

All statistic characteristics of the flow are equal and independent of direction
→ Transitional and rotational invariance

Turbulence Models

Turbulence models may be characterized by the number of PDEs they additionally solve.

- Zero equation model: mixing length model.
- One equation model: Spalart-Almaras.
- Two equation models:
  - *k*-ε style models
    - standard,
    - Renormalization group model (RNG),
    - Realizable
  - *k*-ω model
  - Algebraic stress model (ASM)
- Seven equation model: Reynolds stress model

Large Eddy Simulation (LES) which is based on filtered equations, large eddies are resolved, the impact of small ones is covered by sub-grid models

There is as well a possibility to characterize turbulence models according to the concepts they apply.

- Eddy viscosity models
- Reynolds stress transport models
- Non-linear Eddy viscosity models
- Direct modeling of the divergence of Reynolds stresses

Eddy viscosity models rely on the Boussinesq hypothesis:

- Experiments showed that turbulence decays unless there is shear in isothermal incompressible flows
- Turbulence was found to increase as the mean rate of deformation increases.
- Boussinesq proposed in 1877 that the Reynolds stresses could be linked to the mean rate of deformation.

Reynolds-averaged Navier-Stokes Equation

Osborne Reynolds: decompose actual velocity into the sum of an averaged velocity and the velocity fluctuation and we can do that for pressure and density

$$u(x,t)=\overline{u}(x)+u'(x,t) \quad p(x,t)=\overline{p}(x)+p'(x,t) \quad \rho(x,t)=\overline{\rho}(x)+\rho'(x,t)$$

General shorter version $o=\overline{o}+o'$

However, any true average requires to integrate over a sufficient time interval and nobody really knows how large T has to be.

$$\overline{u}=\lim_{T\to\infty}\int_0^T u(x,t); \qquad 0=\lim_{T\to\infty}\int_0^T u'(x,t)$$

Averaging the continuity equation yields

$$\overline{\left(\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right)\right)}=0 \qquad \frac{\partial\overline{\rho}}{\partial t}+\frac{\partial}{\partial x_i}\left(\overline{\rho}\,\overline{u}_i+\overline{\rho' u_i'}\right)=0;$$

For incompressible flows

$$\frac{\partial}{\partial x_i}\left(\overline{u}_i\right)=0;$$

Reynolds-averaged Navier-Stokes Equation (Osborne Reynolds)

Averaging the momentum equation yields

$$\overline{\frac{\partial}{\partial t}\left(\rho(\bar{u}_j+u'_j\right)}+\overline{\frac{\partial}{\partial x_i}\left(\rho(\bar{u}_i+u'_i)(\bar{u}_j+u'_j\right))}=-\overline{\frac{\partial\left(\bar{p}+p'\right)}{\partial x_j}}+\overline{\frac{\partial}{\partial x_j}}(\bar{\tau}_{ij}+\tau'_{ij})+\rho g_j;$$

Applying averaging rules, the continuity equation

and $\tau'_{ij}=-\rho\overline{u'_i u'_j}$

finally yields

$$\frac{\partial\bar{u}_i}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho\bar{u}_i\bar{u}_j\right)=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij}-\rho\overline{u'_i u'_j}\right)+\rho g_j;$$

Reynolds Stresses

This is where the problem starts. The correlations between the velocity fluctuations are unknown.

You may have heard of the closure problem.

There are a number of different ways to close the problem, named turbulence models

Fluid Mechanic Basics

Favre-averaged Navier-Stokes Equation

For variable density flows (and in particular for combustion) mass-weighted (Favre) averaging is usually preferred to ensemble averaging

Classical time averaging (Reynolds averaging)

$$\overline{\Theta} = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \Theta(\tau) d\tau \quad \text{with } \Theta = \overline{\Theta} + \Theta'$$

Density weighted time averaging (Favre averaging)

$$\widetilde{\Theta} = \frac{1}{\overline{\rho}} \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \rho(\tau)\Theta(\tau) d\tau \quad \text{with } \Theta = \widetilde{\Theta} + \Theta''$$

with Θ some arbitrary variable and Θ‘ and Θ‘‘ the fluctuating parts for Reynolds and Favre averaging.

Continuity

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \left(\overline{\rho}\widetilde{u}_j\right)}{\partial x_j} = 0$$

Momentum

$$\frac{\partial \left(\overline{\rho}\widetilde{u}_j\right)}{\partial t} + \frac{\partial \left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right)}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right)$$

Favre-averaged Navier-Stokes Equation

Energy

$$\frac{\partial \tilde{e}}{\partial t} + \frac{\partial}{\partial x_j}\left[\left(\tilde{e} + \overline{p}\right)\tilde{u}_j\right] = \frac{\partial}{\partial x_j}\left[\left(\overline{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right)\tilde{u}_i\right] - \frac{\partial}{\partial x_j}\left[\overline{q}_j + \overline{\rho u_i'' h''} - \overline{\tau_{ij} u_i''} + \overline{\frac{1}{2}\rho u_j'' u_i'' u_i''}\right]$$

$$\text{with } \overline{\tau}_{ij} = 2\overline{\mu}\overline{S}_{ij};\ \overline{\mu} = \mu_{ref}\left(\frac{\rho_{ref}}{p_{ref}}\frac{\overline{p}}{\overline{\rho}}\right)^{3/4};\ \overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x} - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right);\ \overline{q}_j = -\frac{\kappa}{\kappa - 1}\frac{\overline{\mu}}{\Pr}\left(\frac{\overline{p}}{\overline{\rho}}\right)$$

To close the system we need an equation of state
Assuming a perfect gas we may derive

$$\overline{p} = (\kappa - 1)\left[\tilde{e} - \frac{1}{2}\overline{\rho}\tilde{u}_j\tilde{u}_j - \overline{\rho}k\right] \quad \text{with} \quad k = \frac{1}{2}\overline{u_i'' u_i''}$$

Comparison of Reynolds-averaged and Favre-averaged Navier-Stokes Equation for compressible flows

Continuity

Reynolds-averaged

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\overline{u}_i + \overline{\rho' u_i'}\right) = 0;$$

Favre-averaged

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\widetilde{u}_i\right) = 0;$$

Momentum

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\overline{u}_i\overline{u}_j\right) = -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \overline{\rho u_i' u_j'}\right);$$

$$\frac{\partial}{\partial t}\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right) + \frac{\partial}{\partial x_i}\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right);$$

Turbulence Models

The Basis for all **Eddy Viscosity Models** is the Boussinesq Relation:

$$R_{ij} = \overline{-u_i' u_j'} = 2\frac{\mu_t}{\rho} S_{ij}; \quad \text{with} \quad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

μ_t Eddy viscosity or turbulent viscosity

$$\nu_t = \frac{\mu_t}{\rho}$$

ν_t kinematic turbulent viscosity

Hence, we have the viscous stresses ……

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

…… and the Reynolds stresses

$$\tau_{ij}' = -\rho\overline{u_i' u_j'} = \mu_t\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

…… which finally yields the Reynolds-averaged momentum equation

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{(\mu + \mu_t)}{\rho}\frac{\partial \overline{u}_j}{\partial x_j}\right]$$

Eddy Viscosity Models

$\boldsymbol{\mu_t}$ is non-homogeneous but assumed isotropic which may not always be the case (i.e. flows with strong circulation or swirl)

Mixing Length Model (zero equation)

Dimensional reasoning leads to the argument the turbulent viscosity as the product of a velocity scale and a length scale and assuming that the velocity scale is proportional to a length scale and the gradient of the velocity scale we end up at Prandtl's mixing length model

$$\nu_t = l_m \left| \frac{\partial u}{\partial y} \right|$$

Pros
- Easy to apply and low cost
- Decent results for simple flows with experimental correlations for the mixing length

Cons:
- No capability to predict flows with varying turbulent length scales (recirculation, swirl)
- Provides only mean flow properties and turbulent shear stress.
- Purely local, no history effect

Eddy Viscosity Models

Spalart-Allmaras (one equation)

Solves single conservation equation for turbulent viscosity which contains convective and diffusive transport terms as terms for production and dissipation of turbulent viscosity. Model has been frequently used for aeronautic applications

$$\frac{\partial \bar{\nu}}{\partial t} + \bar{u}_j \frac{\partial \bar{\nu}}{\partial x_j} = C_{b1} \bar{S} \bar{\nu} + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left((\nu + \bar{\nu})\frac{\partial \bar{\nu}}{\partial x_j}\right) + C_{b2}\frac{\partial \bar{\nu}}{\partial x_j}\frac{\partial \bar{\nu}}{\partial x_j}\right] - C_{w1} f_w \left(\frac{\bar{\nu}}{d}\right)^2$$

Production *Diffusion* *Dissipation*

$$\nu_t = \bar{\nu} f_{v1}; \quad f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}^3}; \quad \chi = \frac{\bar{\nu}}{\nu} \qquad \bar{S} = \sqrt{2\Omega_{ij}\Omega_{ij}}\, f_{v3} + \frac{\bar{\nu}}{k^2 d^2} f_{v2} \qquad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Eddy Viscosity Models

Spalart-Allmaras (one equation)

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}; \qquad f_{v3} = 1; \quad f_w(g) = g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6} \qquad g = r + c_{w2}(r^6 - r); \quad r = \frac{\overline{\nu}}{\overline{S}k^2 d^2}$$

Values for constants in standard model

C_{b1}	C_{b2}	σ	C_{v1}	C_{v2}	C_{w3}
0.1355	0.622	2/3	7.1	0.3	2

Pros

- Easy to apply and low cost
- Decent results for simple flows with experimental correlations for the mixing length

Cons:

- No capability to predict flows with varying turbulent length scales (recirculation, swirl)
- Provides only mean flow properties and turbulent shear stress.
- Purely local, no history effect

Eddy Viscosity Models

k-*ε* Model (two equation)

These models focus on mechanisms which affect turbulent kinetic energy *k* of a flow.

The instantaneous kinetic energy *k(t)* is the sum of the mean kinetic energy and turbulent kinetic energy.

$$k(t) = \bar{k} + k'; \quad \bar{k} = \frac{1}{2}\sum_{i=1}^{3}\bar{u}_i; \quad k' = \frac{1}{2}\sum_{i=1}^{3}\overline{u_i'^2}$$

ε is the dissipation rate of *k*

With known *k* and *ε* the turbulent viscosity can be modeled to: $\nu_t \propto k^{1/2}\frac{k^{3/2}}{\varepsilon} = \frac{k^2}{\varepsilon}$

Hence, we need equations for *k* and *ε*

Eddy Viscosity Models

k-*ε* Model (two equation)

Transport equation for turbulent kinetic energy *k*

$$\frac{\partial \bar{k}}{\partial t} + \bar{u}_j \frac{\partial \bar{k}}{\partial x_j} = \frac{\mu_t}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\mu}{\rho}\overline{\frac{\partial u_i'^2}{\partial x_k^2}} + \frac{\partial}{\partial x_j}\left[\frac{\mu}{\rho}\frac{\partial \bar{k}}{\partial x_j} - \frac{1}{2}\overline{u_i' u_j' u_k'} - \overline{p' u_j'}\right]$$

$$\frac{1}{2}\overline{u_i' u_j' u_k'} + \overline{p' u_j'} \approx -\frac{\mu_t}{\rho \sigma_k}\frac{\partial \bar{k}}{\partial x_j}$$

$$\frac{\partial \bar{k}}{\partial t} + \bar{u}_j \frac{\partial \bar{k}}{\partial x_j} = \frac{\mu_t}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\mu}{\rho}\overline{\frac{\partial u_i'^2}{\partial x_k^2}} + \frac{\partial}{\partial x_j}\left[\frac{\mu}{\rho}\frac{\partial \bar{k}}{\partial x_j} - \frac{\mu_\tau}{\rho \sigma_k}\frac{\partial k}{\partial x_j}\right]$$

Convection *Production* *Dissipation* *Viscous Diffusion* *Turbulent Diffusion*

Eddy Viscosity Models

k-*ε* Model (two equation)

Transport equation for *ε*

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}\left(C_{1\varepsilon} \frac{\mu_\tau}{\rho}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right)\frac{\partial \overline{u}_i}{\partial x_j} - C_{2\varepsilon}\varepsilon \right) + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right)\frac{\partial \varepsilon}{\partial x_j} \right] \qquad \text{with } \mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

We have then 5 constants: σ_k, σ_ε, $C_{1\varepsilon}$, $C_{2\varepsilon}$, C_μ which can be obtained investigating simple flows such as decaying homogeneous isotropic turbulence, homogeneous shear flow, the logarithmic wall layer or by comparison with experimental data.

Eddy Viscosity Models

k-ε Model (two equation)

$$\frac{\partial \bar{k}}{\partial t} + \bar{u}_j \frac{\partial \bar{k}}{\partial x_j} = \frac{\mu_t}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\mu}{\rho}\overline{\frac{\partial u_i'^2}{\partial x_k^2}} + \frac{\partial}{\partial x_j}\left[\frac{\mu}{\rho}\frac{\partial \bar{k}}{\partial x_j} - \frac{\mu_\tau}{\rho \sigma_k}\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}\left(C_{1\varepsilon}\frac{\mu_\tau}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - C_{2\varepsilon}\varepsilon\right) + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

The most commonly used values for these constants are:
$\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$, $C_{1\varepsilon} = 1.44$, $C_{2\varepsilon} = 1.92$, $C_\mu = 0.09$.

Assumptions:

$$0 = \lim_{T\to\infty} \int_0^T u'(x,t)$$

- Reynolds averaging: what is the appropriate integration time *T*
- Boussinesq hypothesis: Reynolds stresses linearly dependent on mean strain rate
- Turbulence is homogenous and isotropic
- Equilibrium turbulence (rate of viscous dissipation is balanced by the rate of production of turbulent kinetic energy

However, real fluids are sometime non-homogenous and anisotropic.

Fluid Mechanic Basics

Eddy Viscosity Models

RANS models require a certain way to treat wall-bound flows properly because the local Reynold numbers are small, there are high shear rates, a tendency for two-component turbulence and wall blocking effect of the pressure field

- Wall function approach which aims at bridging viscous region with analytical functions)
- Specific model modifications of the original formulation
- Elliptic relaxation model for pressure-strain correlation

$$\nu_t = f_\mu C_\mu \frac{k^2}{\varepsilon} \quad (0 \le f_\mu(y) \le 1) \qquad f_\mu(y) = 1 - e^{\left(-0.002\, y^+ - 0.00065\, y^{+2}\right)}$$ (Rodi & Mansour)

$$\nu_t = C'_\mu u'_j u'_j \frac{k}{\varepsilon} \quad with \quad C'_\mu = 0.22$$ (Durbin)

Eddy Viscosity Models

RANS models require a certain way to treat wall-bound flows properly because the local Reynold numbers are small, there are high shear rates, a tendency for two-component turbulence and wall blocking effect of the pressure field

- Wall function approach which aims at bridging viscous region with analytical functions
- Specific model modifications of the original formulation
- Elliptic relaxation model for pressure-strain correlation

$$\nu_t = f_\mu C_\mu \frac{k^2}{\varepsilon} \quad (0 \le f_\mu(y) \le 1) \qquad f_\mu(y) = 1 - e^{\left(-0.002\, y^+ - 0.00065\, y^{+2}\right)}$$ (Rodi & Mansour)

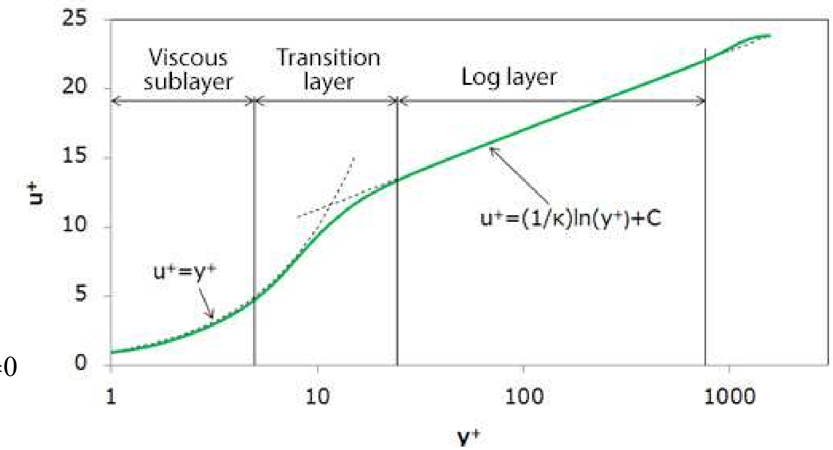
$$\nu_t = C'_\mu u'_j u'_j \frac{k}{\varepsilon} \quad with \quad C'_\mu = 0.22$$ (Durbin)

Most near-wall models are not reliable

Fluid Mechanic Basics

Eddy Viscosity Models

k-*ε* Model (two equation)



Solving the near-wall problem by

- Implementation of a wall function, i.e. universal velocity law
  - first grid point for numerical solution should be put in the log area ($y^+ > 10$)
  - Normally *k* set to 0 at the wall but ε ‡ 0, usually set to $\varepsilon = \frac{\mu}{\rho}\frac{\partial^2 k}{\partial y}\Big|_{y=0}$

- Implementation of damping functions

  k-equation unchanged

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}\left(f_1 C_{1\varepsilon} \frac{\mu_\tau}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - f_2 C_{2\varepsilon}\varepsilon \right) + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \quad \text{with} \quad \mu_t = \rho f_u C_\mu \frac{k^2}{\varepsilon}$$

Classic Launder Sharma model:

$$f_1 = 1;\ f_2 = 1 - 0.3 e^{\mathrm{Re}_t^2};\ f_u = \frac{-3.4}{e^{(1+0.02\mathrm{Re}_t)^2}};\ \mathrm{Re}_t = \frac{\rho k^2}{\mu \varepsilon}$$

Eddy Viscosity Models

k-ε Model

Advantages

- Easy to apply and low cost
- Stable calculations, easy convergence
- Reasonable results for many flows

Disadvantages

- Poor predictions for
  - Swirling and rotating flows
  - Flows with strong separation and recirculation
  - Axisymmetric jets
  - Some unconfined flows
  - Flows in non-circular ducts
- Valid only for fully turbulent flows
- Simplistic *ε*-equation

There are many 2-equation models which aim at solving some of these shortcomings

- *k-ε* RNG (Renormalization Group): *k-ε* equations derived applying the RNG method lead to standard *k-ε* equations but include a term which
  - accounts for the interaction between turbulence dissipation and mean shear,
  - Accounts for the effect of swirl on turbulence,
  - has an analytical formula for the turbulent Prandtl number and a
  - differential form of the effective viscosity

  and thus solves most of the issues mentioned above but problem of free jet spreading still present

Eddy Viscosity Models

k-ε RNG Model

The k equation is identical

$$\frac{\partial \bar{k}}{\partial t} + \bar{u}_j \frac{\partial \bar{k}}{\partial x_j} = \frac{\mu_t}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\mu}{\rho}\frac{\overline{\partial u_i'^2}}{\partial x_k^2} + \frac{\partial}{\partial x_j}\left[\frac{\mu}{\rho}\frac{\partial \bar{k}}{\partial x_j} - \frac{\mu_\tau}{\rho\sigma_k}\frac{\partial \bar{k}}{\partial x_j}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}\left(C_{1\varepsilon}\frac{\mu_\tau}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} - \boxed{C_{2\varepsilon}^{*}\varepsilon}\right) + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

$$C_{2\varepsilon}^{*} = C_{2\varepsilon} + \frac{C_\mu \eta^3 (1-\eta/\eta_0)}{1+\beta\eta^3}; \ \eta = \frac{\bar{k}}{\varepsilon}\left(\frac{\partial \bar{u}_{\mathrm{i}}}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j};$$

The standard model constants (in brackets) are modified:

σ_k = 0.7194 (1.0), σ_ε = 0.7194 (1.3), $C_{1\varepsilon}$ = 1.42 (1.44), $C_{2\varepsilon}$ = 1.62 (1.92), C_μ = 0.0845 (0.09), η_0 = 4.38, β = 0.012

Eddy Viscosity Models

SST k-ω Model (Mentor and Wilcox)

Kinematic Eddy Viscosity

$$\mu_\tau = \rho \frac{a_1 \bar{k}}{\max(a_1\omega, SF_2)}$$

Turbulent Kinetic Energy

$$\frac{\partial \bar{k}}{\partial t} + \bar{u}_j \frac{\partial \bar{k}}{\partial x_j} = P_k - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\rho} + \sigma_k \frac{\mu_\tau}{\rho}\right)\frac{\partial \bar{k}}{\partial x_j}\right]$$

Specific Dissipation Rate

$$\frac{\partial \omega}{\partial t} + \bar{u}_j \frac{\partial \omega}{\partial x_j} = \alpha S^2 - \beta\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\rho} + \sigma_\omega \frac{\mu_\tau}{\rho}\right)\frac{\partial \omega}{\partial x_j}\right] + 2(1-F_1)\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial \bar{k}}{\partial x_i}\frac{\partial \omega}{\partial x_i}$$

Closure terms

$$F_2 = \tanh\left[\left[\max\left(\frac{2\sqrt{\bar{k}}}{\beta^*\omega y}, \frac{500\frac{\mu}{\rho}}{y^2\omega}\right)\right]^2\right]; F_1 = \tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{\bar{k}}}{\beta^*\omega y}, \frac{500\frac{\mu}{\rho}}{y^2\omega}\right), \frac{4\sigma_{\omega 2}\bar{k}}{CD_{k\omega}y^2}\right]\right\}^4\right\}; \phi = \phi_1 F_1 + \phi_2 F_2;$$

$$P_k = \min\left(\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}, 10\beta^*\bar{k}\omega\right); \alpha_1 = \frac{5}{9}, \alpha_2 = 0.44; \beta_1 = \frac{3}{40}, \beta_2 = 0.0828, \beta^* = \frac{9}{100}; \sigma_{k1} = 0.85, \sigma_{k2} = 1, \sigma_{\omega 1} = 0.5, \sigma_{\omega 2} = 0.856$$

Eddy Viscosity Models

Model	**Basic Features**
Spalart-Almaras	Single transport equation model which solves directly a modified turbulent viscosity; best suited for aerospace applications; optional inclusion of strain rate in *k* – equation to improve swirl flow predictions
Standard *k-ε*	Baseline two-equation model solving *k* and *ε* equations; empirical coefficients; comes in many variants;
RNG *k-ε*	Variant with derives equations and coefficients derived analytically in particular for the *ε* equation to improve handling of high strain flows
Realizable *k-ε*	Variant which omits negative normal stresses and violation of the Schwartz shear stress inequality for large strain rates by defining C_μ variable;
SST *k-ω*	Combines original Wilcox model for wall-bounded and low Re flows with a *k-ε* core flow treatment via a blending function;

Eddy Viscosity Models

Model	Behavior / Performance
Spalart-Almaras	Poor performance for 3D flows, free shear flows, flows with recirculation; suitable for aero-applications
Standard k-ε	Poor performance for complex flows; suitable for initial keening of design variations, parametric studies,
RNG k-ε	Suitable for complex flows with large strain, moderate swirl and transitional flows
Realizable k-ε	Similar to RNG model with advantages on accuracy and easier convergence;
k-ω SST	Suited for both wall bounded and core flows,

Crux of Eddy Viscosity Turbulence Models

Assumption of homogeneous, isotropic turbulence doesn't hold for a large number of flow situations.

- i.e. a flat plate boundary layer there is no isotropy, instead:
- The phenomenon of intermittency when the form of the probability distribution changes inside the inertial range, i.e. scaling anomaly

$$\overline{u_i'^2} : \overline{u_j'^2} : \overline{u_k'^2} = 4 : 2 : 3$$

→ Necessity of Means to overcome this problem

→ Reynolds Stress Models

Fluid Mechanic Basics

Reynolds Stress Models (2nd moment closure models)

Transport equation for Reynolds stresses

$$\frac{\partial \tau_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\left(\frac{\partial \overline{u}_j}{\partial x_k} + \frac{\partial \overline{u}_i}{\partial x_k}\right) - \frac{\mu}{\rho}\frac{\partial^2 \tau_{ij}}{\partial x_k^2}$$

$$-\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right) - 2\frac{\mu}{\rho}\overline{\frac{\partial u_i' \partial u_j'}{\partial x_k \partial x_k}} + \overline{p'\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)} - \frac{\partial}{\partial x_k}\left(\frac{\overline{p' u_j'}}{\rho}\delta_{jk} + \frac{\overline{p' u_i'}}{\rho}\delta_{ik}\right)$$

This equation can be read as:

rate of change of stresses, plus transport of stresses due to convection $\frac{\partial \tau_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}}{\partial x_k}$ equals production $-\tau_{ik}\left(\frac{\partial \overline{u}_j}{\partial x_k} + \frac{\partial \overline{u}_i}{\partial x_k}\right)$

viscous diffusion $-\frac{\mu}{\rho}\frac{\partial^2 \tau_{ij}}{\partial x_k^2}$, turbulent transport $-\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)$, dissipation $-2\frac{\mu}{\rho}\overline{\frac{\partial u_i' \partial u_j'}{\partial x_k \partial x_k}}$

Reynolds Stress Models (2nd moment closure models)

pressure strain $\overline{p'\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)}$, pressure transport $\frac{\partial}{\partial x_k}\left(\frac{\overline{p'u'_j}}{\rho}\delta_{jk}+\frac{\overline{p'u'_i}}{\rho}\delta_{ik}\right)$

While the production term is exact, all other terms are usually modelled

Transport: exact: $-\frac{\mu}{\rho}\frac{\partial^2\tau_{ij}}{\partial x_k^2}-\frac{\partial}{\partial x_k}\left(\overline{u'_i u'_j u'_k}\right)+\frac{\partial}{\partial x_k}\left(\frac{\overline{p'u'_j}}{\rho}\delta_{jk}+\frac{\overline{p'u'_i}}{\rho}\delta_{ik}\right)$; model: $\frac{\partial}{\partial x_m}\left(\frac{\nu_\tau}{\sigma_k}\frac{\partial\tau_{ij}}{\partial x_m}\right)$

Dissipation: exact: $-2\frac{\mu}{\rho}\overline{\frac{\partial u'_i\partial u'_j}{\partial x_k\partial x_k}}$; model: $\frac{2}{3}\varepsilon\delta_{ij}$

Pressure strain: exact: $\overline{p'\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)}$; model: $-C_1\frac{\varepsilon}{k}\left(\tau_{ij}-\frac{2}{3}k\delta_{ij}\right)-C_2\left(-\tau_{ij}\left(\frac{\partial\bar{u}_j}{\partial x_k}+\frac{\partial\bar{u}_i}{\partial x_k}\right)-\frac{2}{3}\left(-\tau_{ij}\left(\frac{\partial\bar{u}_j}{\partial x_k}+\frac{\partial\bar{u}_i}{\partial x_k}\right)\delta_{ij}\right)\right)$

Fluid Mechanic Basics

Reynolds Stress Models

$$\frac{\partial \overline{u_i'u_j'}}{\partial t}+\bar{u}_k\frac{\partial \overline{u_i'u_j'}}{\partial x_k}=-\tau_{ik}\left(\frac{\partial \bar{u}_j}{\partial x_k}+\frac{\partial \bar{u}_i}{\partial x_k}\right)-\frac{\mu}{\rho}\frac{\partial^2\tau_{ij}}{\partial x_k^2}-\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right)-2\frac{\mu}{\rho}\frac{\overline{\partial u_i'\partial u_j'}}{\partial x_k\partial x_k}+\overline{p'\left(\frac{\partial u_j'}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right)}-\frac{\partial}{\partial x_k}\left(\frac{\overline{p'u_j'}}{\rho}\delta_{jk}+\frac{\overline{p'u_i'}}{\rho}\delta_{ik}\right)$$

Let’s rearrange it to get a better insight into the meaning of the key terms

$$\frac{\partial \overline{u_i'u_j'}}{\partial t}+\bar{u}_k\frac{\partial \overline{u_i'u_j'}}{\partial x_k}+\underbrace{\frac{\partial}{\partial x_k}\left(\left(\overline{u_i'u_j'u_k'}\right)+\frac{\mu}{\rho}\frac{\partial^2\tau_{ij}}{\partial x_k^2}+\frac{\overline{p'u_j'}}{\rho}\delta_{jk}+\frac{\overline{p'u_i'}}{\rho}\delta_{ik}\right)}_{\text{turbulent transport}}=\underbrace{\tau_{ik}\left(\frac{\partial \bar{u}_j}{\partial x_k}+\frac{\partial \bar{u}_i}{\partial x_k}\right)}_{\text{production}}-\underbrace{2\frac{\mu}{\rho}\frac{\overline{\partial u_i'\partial u_j'}}{\partial x_k\partial x_k}}_{\text{dissipation}}+\underbrace{\overline{p'\left(\frac{\partial u_j'}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right)}}_{\text{pressure-strain}}$$

turbulent transport

production

dissipation

pressure-strain

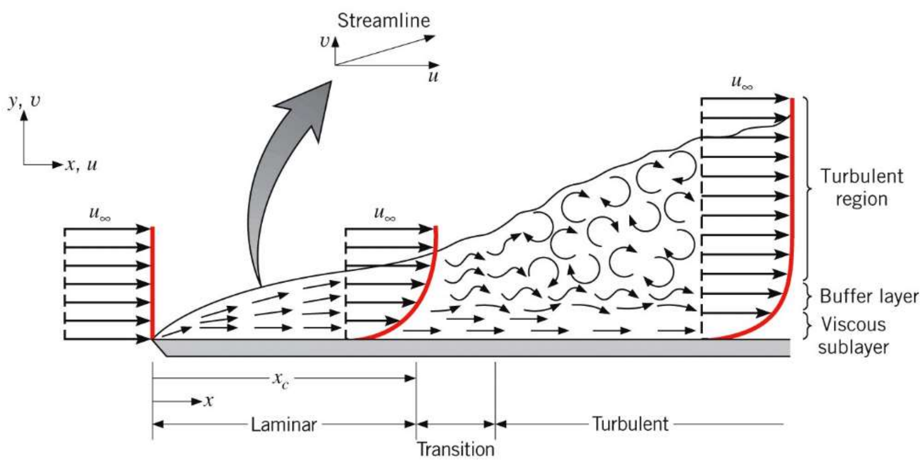
Reynolds Stress Models

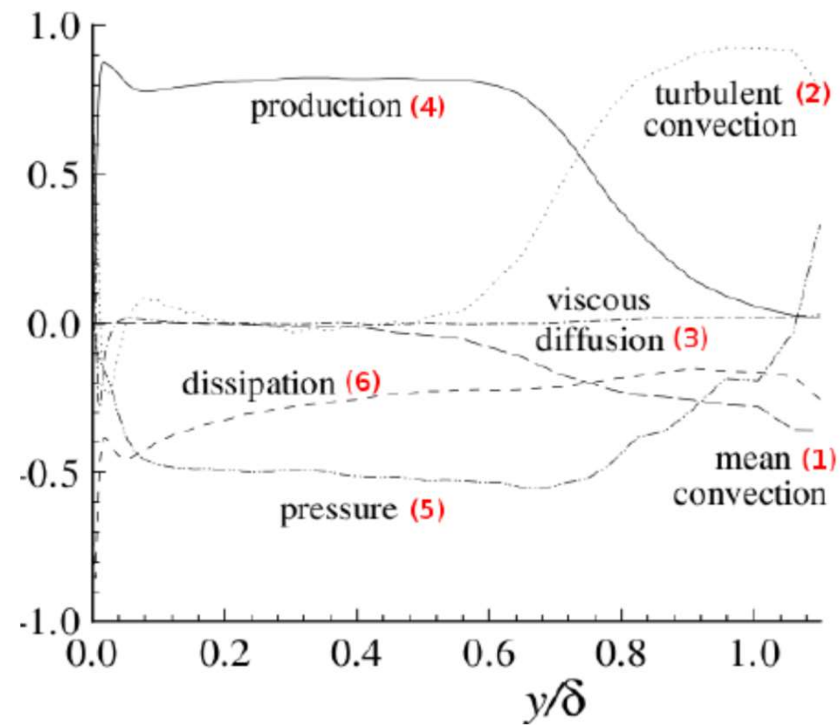
Rearrange equation, abbreviate terms and look at their behavior in a turbulent boundary layer

$$-\left(\frac{\partial \overline{u_i'u_j'}}{\partial t}+\bar{u}_k\frac{\partial \overline{u_i'u_j'}}{\partial x_k}\right)-\frac{\partial}{\partial x_k}\left(u_i'u_j'u_k'\right)+\nu\frac{\partial^2}{\partial x_k^2}\left(u_i'u_j'\right)+\Pi_{ij}+P_{ij}-\varepsilon_{ij}=0$$

(1) (2) (3) (4) (5) (6)

Stream-wise normal stress $\langle u_1'u_1'\rangle$





Production mainly balanced by dissipation and pressure-rate-of-strain terms

Fluid Mechanic Basics

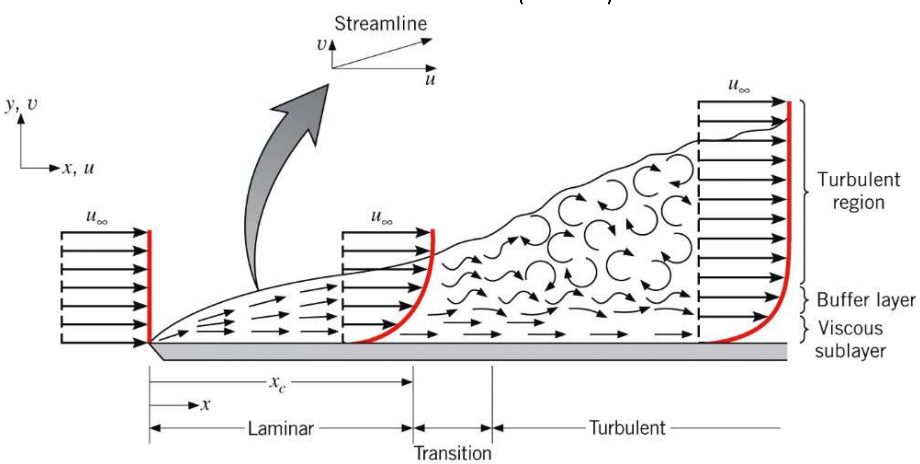
Reynolds Stress Models

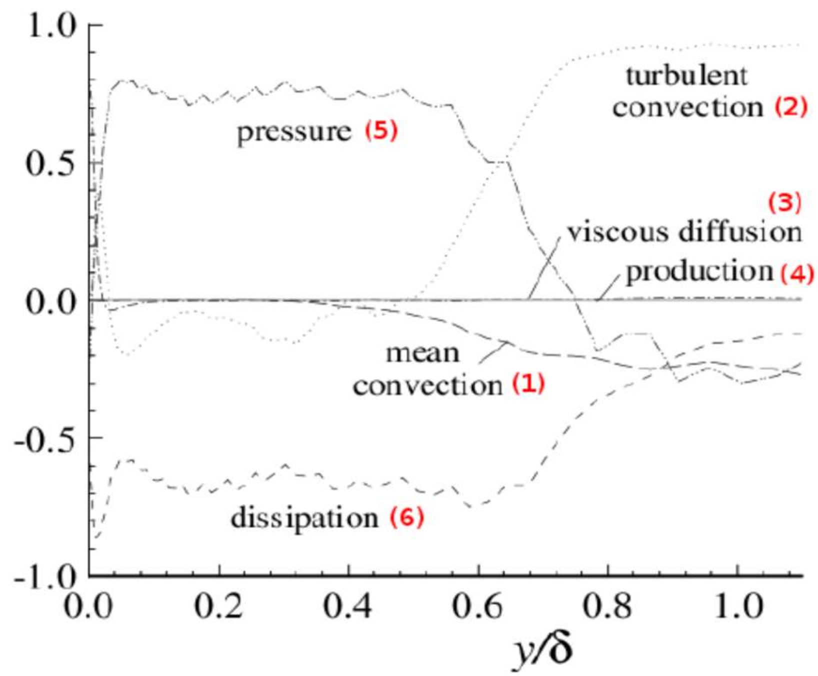
Rearrange equation, abbreviate terms and look at their behavior in a turbulent boundary layer

$$-\left(\frac{\partial \overline{u_i' u_j'}}{\partial t} + \bar{u}_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(u_i' u_j' u_k'\right) + \nu \frac{\partial^2}{\partial x_k^2}\left(u_i' u_j'\right) + \Pi_{ij} + P_{ij} - \varepsilon_{ij} = 0$$

(1) (2) (3) (4) (5) (6)

Wall-normal normal stress $\langle u_2' u_2' \rangle$



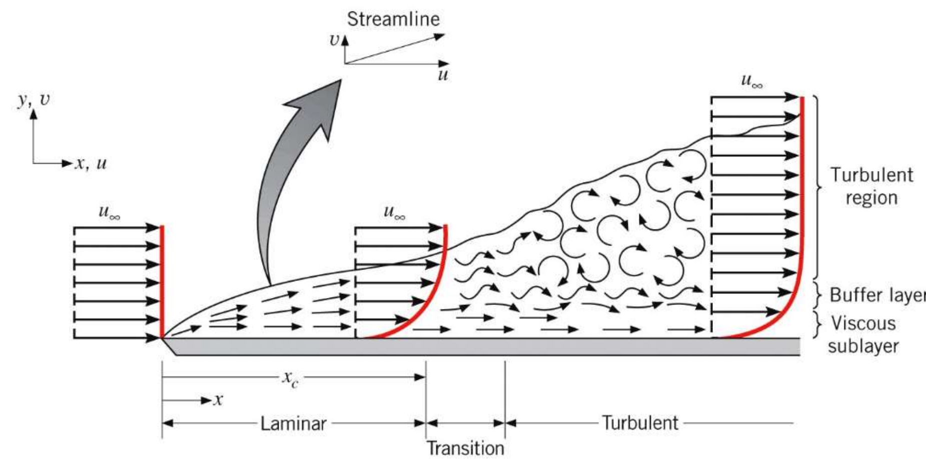


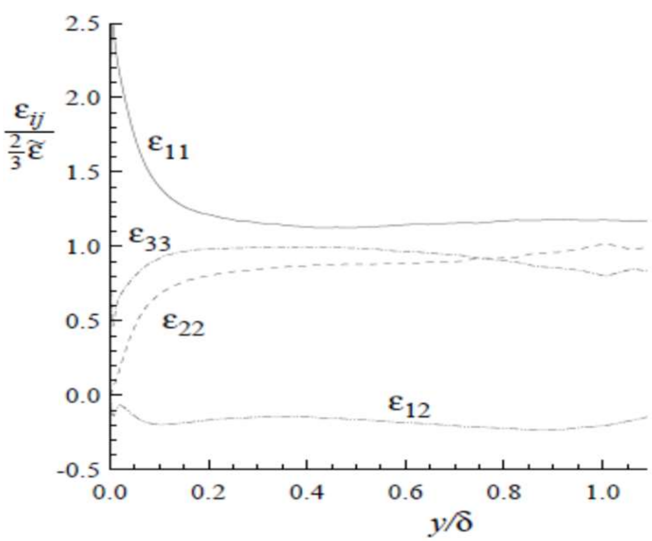
- No production
- Gain from pressure-rate-of-strain term balanced by dissipation

Reynolds Stress Models

Rearrange equation, abbreviate terms and look at their behavior in a boundary layer

Dissipation ε_{ij}





(DNS Spalart 1988, $Re_\theta = 1410$)

From observations:

- For high Re dissipation tensor almost isotropic
- For low Re some residual anisotropy but significant anisotropy near the wall

Quite often dissipation is modelled as isotropic to:

$$\varepsilon_{ij} = \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

with $\tilde{\varepsilon}$ the scalar dissipation rate

Reynolds Stress Models

Rearrange equation, abbreviate terms and look at their behavior in a boundary layer

Pressure-rate-of strain tensor $$R_{ij} = \overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)}$$

From observations:

- Pressure terms are **significant and difficult** to model
- Pressure rate of strain correlation has a **redistributive** character due to the interactions among the fluctuating velocity field and the mean velocity gradient field

Decomposition of pressure fluctuations into a rapid, slow and harmonic part: $$p' = p^R + p^S + p^H$$

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left(p'^R + p'^S + p'^H\right) = -2\frac{\partial \overline{u}_i \partial u'_j}{\partial x_j \partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}\left(u'_i u'_j - \overline{u'_i u'_j}\right)$$

$\left|\frac{\partial \overline{u}_i}{\partial x_j}\right| \to \infty$ rapid $$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left(p'^R\right) = -2\frac{\partial \overline{u}_i \partial u'_j}{\partial x_j \partial x_i}$$

harmonic $$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left(p'^H\right) = 0$$

$\left|\frac{\partial \overline{u}_i}{\partial x_j}\right| \to 0$ slow $$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_j}\left(p'^S\right) = -\frac{\partial^2}{\partial x_j \partial x_i}\left(u'_i u'_j - \overline{u'_i u'_j}\right)$$

only important near walls

Fluid Mechanic Basics

Reynolds Stress Models

Pressure-rate-of strain tensor $R_{ij} = R_{ij}^R + R_{ij}^S + R_{ij}^H$

$$R_{ij}^R = \overline{\frac{p'^R}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)} \qquad R_{ij}^S = \overline{\frac{p'^S}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)} \qquad R_{ij}^H = \overline{\frac{p'^S}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)}$$

"Slow" Pressure-rate-of strain for homogenous decaying turbulence but with anisotropy

$$\frac{\partial \overline{u'_i u'_j}}{\partial t} = -\varepsilon_{ij} + \Pi_{ij}$$

All other terms vanish due to homogenous decaying turbulence assumption ⇒

$$\frac{\partial \overline{u'_i u'_j}}{\partial t} = R_{ij}^S - \varepsilon_{ij}$$

Simple Reynolds stress equation for homogenous turbulence

Model: $R_{ij}^S = \varepsilon_{ij} F_{ij}^S(b_{ij})$ with $F_{ij}^S = C_1 b_{ij} + C_2\left(b_{ij}^2 - \frac{1}{3} b_{kk}^2 \delta_{ij}\right)$ $\quad b_{ij} = \frac{\overline{u_i u_j}}{2k} - \frac{\delta_{ij}}{3}$, Reynolds stress anisotropy tensor

Rotta's assumption: $C_1 = -2\,C_R$, $C_2 = 0$ $\qquad R_{ij}^S = -2 C_R \frac{\tilde{\varepsilon}}{k}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}\right)$ linear return to isotropy

However, this model doesn't work in all cases due to missing non-linearity.

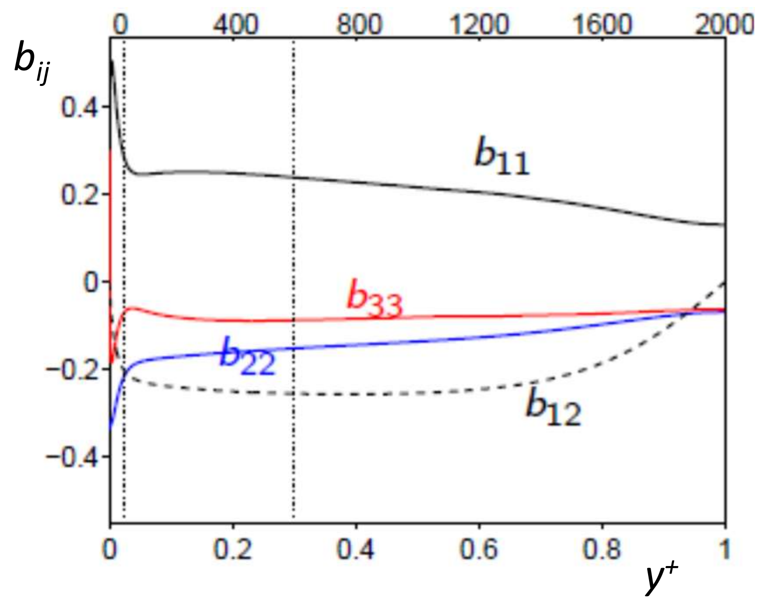
Reynolds Stress Models

"Slow" Pressure-rate-of strain for homogenous decaying turbulence but with anisotropy

$$b_{ij} = \frac{\overline{u_i u_j}}{2k} - \frac{\delta_{ij}}{3},$$

Reynolds stress anisotropy tensor

- b_{ij} shows strong changes in viscous wall region
- Is almost constant in log-region
- Tends towards isotropy in core flow



Speziale model: ***$C_2 \neq 0$***

Fluid Mechanic Basics

Reynolds Stress Models

“Rapid” Pressure-rate-of strain

$$\frac{\partial \overline{u'_i u'_j}}{\partial t} = P_{ij} + R_{ij} - \frac{2}{3}\tilde{\varepsilon}\delta_{ij}$$

$$R_{ij}^{R} = 2\frac{\partial u_l}{\partial x_k}\left(\mathrm{M}_{kjil} + \mathrm{M}_{ikjl}\right)$$

- M_{ijkl} is an integral of two-point correlations
- in single-point closures M_{iljk} is modelled as function of $(b_{ij,} k)$
- Symmetry, tensorial considerations & realizability constraints lead to

$$R_{ij}^{R} = k\sum_{n=3}^{8} f(n)\mathrm{T}_{ij}(n)$$

Fluid Mechanic Basics

Reynolds Stress Models

Common Rapid Pressure Strain Models

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right);$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

- LRR: $C_2 = 0.4$
- SL: $C_2 = \frac{1}{10}\left(1 + \frac{4}{5}(g(b_{ij}b_{ji}, b_{ij}b_{jk}b_{ki})\right)$

n	$T_{ij}(n)$	f(n)		
		LRR (Launder et al.)	SSG (Speziale et al.)	SL (Shih & Lumley)
3	S_{ij}	4/5	$4/5 + 1.3(-1/2 b_{ij} b_{ji})^{1/2}$	4/5
4	$b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{lk}S_{kl}\delta_{ij}$	$6/11(2 + 3C_2)$	5/4	$12\ C_2$
5	$\Omega_{ik}b_{jk} - b_{ik}\Omega_{kj}$	$2/11(10 - 7C_2)$	2/5	$4/3(2 - 7C_2)$
6	$b_{ik}^2 S_{jk} + b_{kj}^2 S_{ik} - \frac{2}{3}b_{lk}^2 S_{lk}\delta_{ij}$	0	0	4/5
7	$\Omega_{ik}b_{jk}^2 - b_{ik}^2\Omega_{kj}$	0	0	4/5
8	$b_{jk}S_{kl}b_{lj} - \frac{1}{3}b_{lk}^2 S_{kl}\delta_{ij}$	0	0	-8/5
Satisfaction of realizability		no	yes	yes

Large Eddy Simulation (LES)

Basics

- consider the time-dependent Navier-Stokes equations
- resolve only the large scales of motion numerically
- replace the action of the small scales by a model (sub-grid model)

Approach

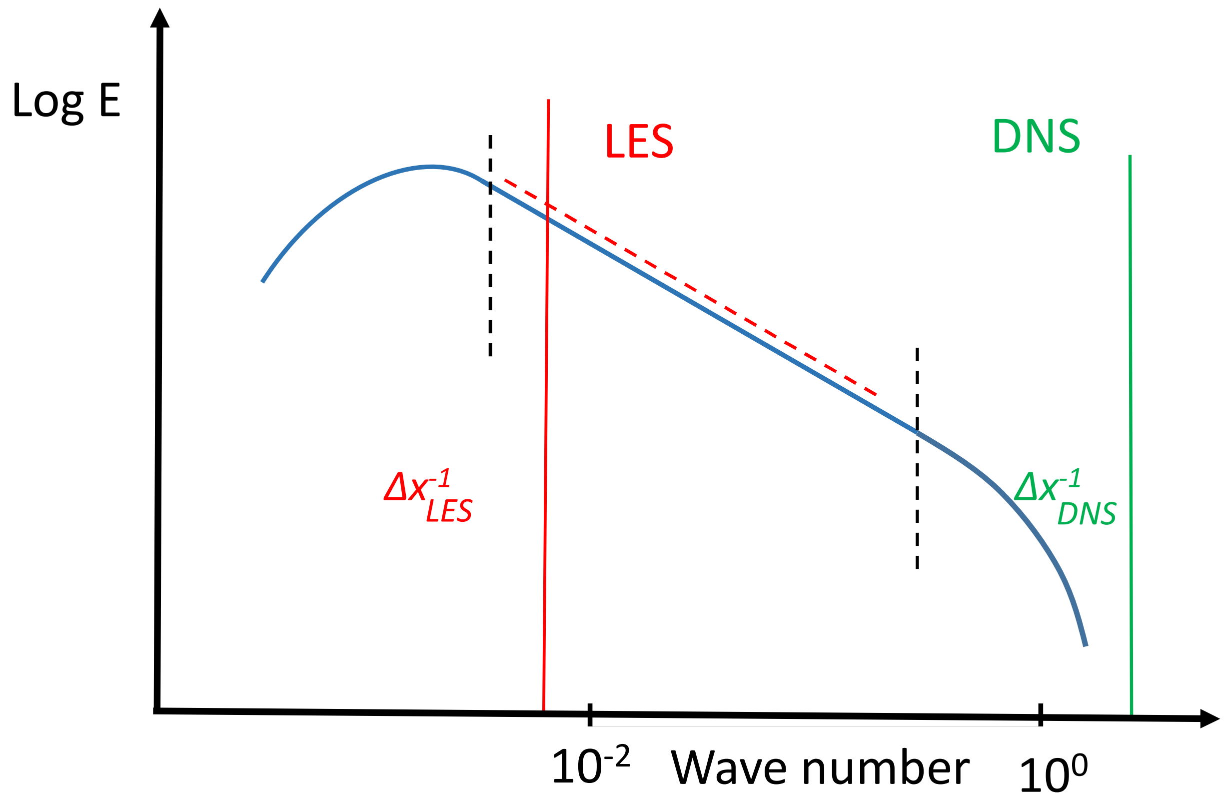
- define a spatial filter
- derive the filtered Navier-Stokes equations
- choose a model for the unclosed subgrid-stress term
- solve the closed equations numerically

→ Approximation of large-scale motions

Large Eddy Simulation (LES)

Filtering can be done explicitly (application of a filter) or implicitly (numerical grid).

1. Explicit filtering: Spatial filtering, residual stress modeling and numerical solution of model equation are treated separately.
2. Implicit filtering: numerical discretization errors are deliberately treated as part of model



Large Eddy Simulation

Definition of Filter Operation

General definition

$$\overline{\mathbf{u}}(x,t)=\int \underbrace{G(\mathbf{r},x)}_{\text{filter fct.}}\underbrace{\mathbf{u}(x-\mathbf{r},t)}_{\text{signal}}d\mathbf{r} \quad \text{(volume integral)}$$

Normalization condition: $\int G(\mathbf{r},x)\,d\mathbf{r}=1 \quad \forall x$

Decomposition: $\mathbf{u}(\mathbf{x},t)=\underbrace{\overline{\mathbf{u}}(\mathbf{x},t)}_{\text{filtered}}+\underbrace{\mathbf{u}'(\mathbf{x},t)}_{\text{residual}}$

This looks like Reynolds decomposition, however:

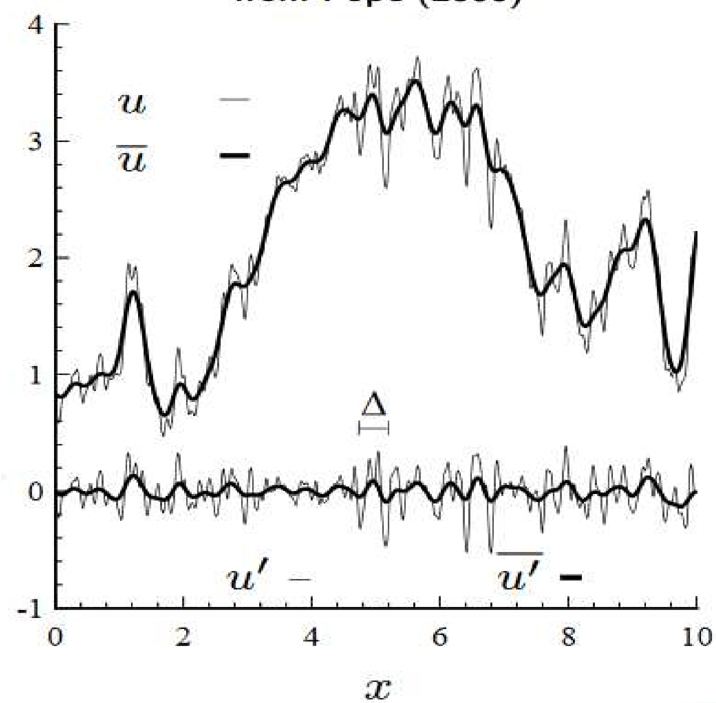
$\overline{\mathbf{u}}(\mathbf{x},t)$ is a random field !
And in addition: $\overline{\mathbf{u}'}(\mathbf{x},t)\neq 0$ and $\overline{\overline{\mathbf{u}}}(\mathbf{x},t)\neq \overline{\mathbf{u}}(\mathbf{x},t)$

1D spatial case, homogenous filter function G(r)

$$\overline{u}(x)=\int_{-\infty}^{\infty} G(r)\,u(x-r)\,dr$$

- box filter with Δ =0.35
- filtered signal $\overline{u}$ smoother
- filtered residual $\overline{u'}\neq 0$
- Repeated filtering increases smoothness

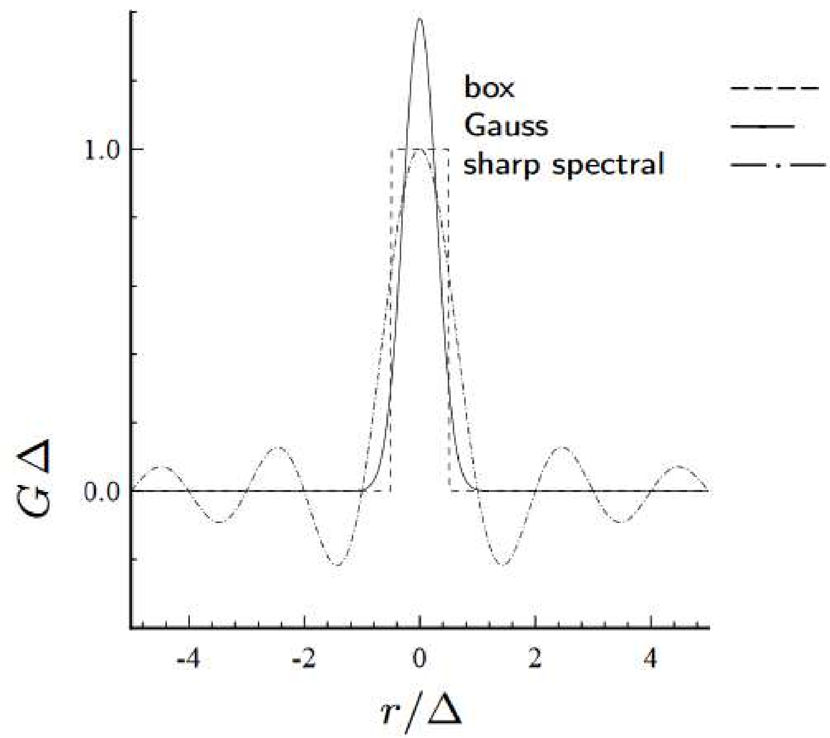
from Pope (2000)



Common Filter Function in Physical Space

	$G(r)$
Box	$\frac{1}{\Delta} H\left(\frac{1}{2}\Delta - \lvert r \rvert\right)$
Gauss	$\left(\frac{6}{\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{6r^2}{\Delta^2}\right)$
Sharp spectral	$\sin\left(\frac{\pi r}{\Delta}\right) / (\pi r)$

from Pope (2000)



Large Eddy Simulation

Filter in Spectral Space

Filtering removes high frequency scale and can be interpreted in Fourier space.

Fourier transformation of a scalar field $\phi(x,t) \rightarrow \hat{\phi}(k,\omega)$ with k, the spatial wave number and, ω, the temporal frequency. $\hat{\phi}$ will be filtered by the corresponding Fourier transform of the filter kernel, $\hat{G}(k,\omega)$.

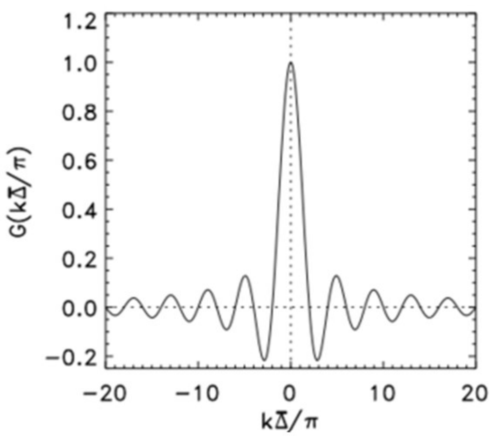
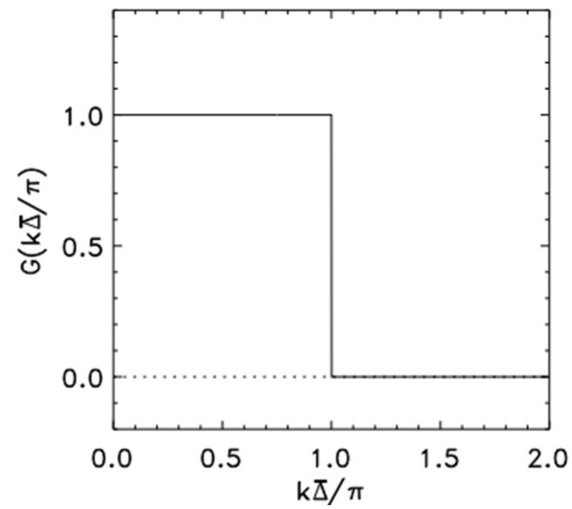
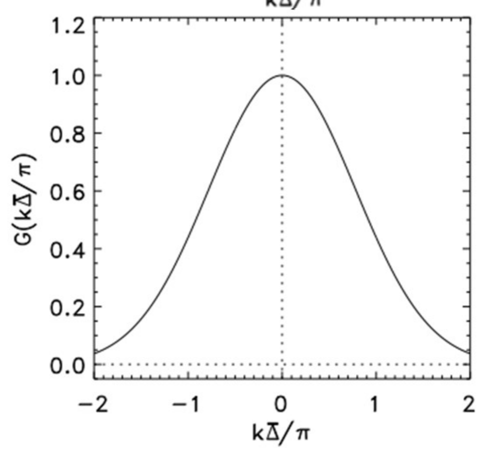
$$\overline{\hat{\phi}}(k,\omega) = \hat{\phi}(k,\omega)\hat{G}(k,\omega)$$

The width of the filter Δ has an associated cutoff wave number $k_{c,}$ the temporal filter width τ_c an associated cutoff frequency ω_c.

The spectral interpretation of the filtering is essential in large eddy simulation (energy cascade → transfer of energy from low to high frequency eddies (sub-grid scale models).

Large Eddy Simulation

Transfer Functions of Common Filters Spectral Space

	$\hat{G}(k)$			$\hat{G}(k)$	
Box	$\sin(k\Delta/2)/(k\Delta/2)$		Sharp spectral	$H(k_c - \lvert k \rvert)$; $k_c = \pi/\Delta$	
Gauss	$\exp\left(-\frac{k^2\Delta^2}{24}\right)$				

Large Eddy Simulation

Properties of Fourier Space Filtering

Filtered field: $\hat{\bar{u}}(k) = \hat{G}(k) \quad \hat{u}(k)$

\`low pass'

Residual field: $\hat{u}'(k) = \left(1 - \hat{G}(k)\right) \quad \hat{u}(k)$

\`high pass'

Double filtered field: $\hat{\bar{\bar{u}}}(k) = \left(\hat{G}(k)\right)^2 \quad \hat{u}(k)$

→ $\hat{\bar{\bar{u}}}(k) = \hat{\bar{u}}$ requires $\left(\hat{G}(x)\right)^2 = \hat{G}(x)$ which is generally not true.

Filtering in 3 D space

3D isotropic filter

$$\bar{u}(\mathbf{x}) = \int_{-\infty}^{\infty} G(|\mathbf{r}|)\, u(x - \mathbf{r})\, d\mathbf{r}$$

3D anisotropic filter

$$\bar{u}(\mathbf{x}) = \int_{-\infty}^{\infty} G(r_1, \Delta_1) G(r_2, \Delta_2) G(r_3, \Delta_3) u(x - \mathbf{r})\, d\mathbf{r}$$

Large Eddy Simulation

Favre Filtering

For LES, Favre filter represents a density-weighted spatial average (in RANS density-weighted averaging)

In general $\overline{\varphi} = \frac{\widetilde{\rho\varphi}}{\widetilde{\rho}} \Rightarrow \widetilde{\rho}\overline{\varphi} = \widetilde{\rho\varphi}$ and in case of incompressible flows: $\overline{\varphi} = \widetilde{\varphi}$

Non-dimensional Favre-filtered <u>continuity equation</u>

$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial\left(\widetilde{\rho}\overline{u}_i\right)}{\partial x_i} = 0$$

Non-dimensional Favre-filtered <u>momentum equation</u>

$$\frac{\partial\left(\widetilde{\rho}\overline{u}_i\right)}{\partial t} + \frac{\partial\left(\widetilde{\rho}\overline{u}_i\overline{u}_j\right)}{\partial x_j} + \frac{\partial \widetilde{p}}{\partial x_i} - \frac{\partial\left(\overline{\sigma}_{ij}\right)}{\partial x_j} = -\frac{\partial\left(\widetilde{\rho}\tau_{ij}\right)}{\partial x_j} + \frac{\partial\left(\widetilde{\sigma}_{ij} - \overline{\sigma}_{ij}\right)}{\partial x_j}$$

With the sub-filter scale (SFS) terms

Smooth viscous stress tensor $\overline{\sigma}_{ij} = \frac{2}{\mathrm{Re}}\overline{\mu}\left(\overline{S}_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \overline{u}_k}{\partial x_k}\right)$ $\qquad \overline{\mu} = f\left(\overline{T}\right)$

Non-linear viscous stress tensor $\widetilde{\sigma}_{ij} = \frac{2}{\mathrm{Re}}\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$ $\qquad \mu = f\left(T\right)$

Large Eddy Simulation

Favre Filtering

Non-dimensional Favre-filtered turbulent kinetic energy equation

$$\frac{\partial(\bar{e})}{\partial t}+\frac{\partial((\bar{e}+\tilde{p})\bar{u}_j)}{\partial x_j}-\frac{\partial(\bar{u}_i\bar{\sigma}_{ij})}{\partial x_j}+\frac{\partial\bar{q}_j}{\partial x_j}=-a_1-a_2-a_3+a_4+a_5-a_6$$

Kinetic energy transferred from resolved scales to SFSs

$$a_1=\bar{u}_i\frac{\partial(\tilde{p}\tau_{ij})}{\partial x_i}$$

$$a_2=\frac{1}{\gamma-1}\frac{(pu_j-\tilde{p}\bar{u}_j)}{\partial x_j}$$

Pressure velocity SFS term (effect of SFS turbulence on heat conduction at resolved scales)

Compressibility effects (vanishes for incompressible flows)

$$a_3=p\frac{\partial u_j}{\partial x_j}-\tilde{p}\frac{\partial\bar{u}_j}{\partial x_j}$$

$$a_4=\sigma_{ij}\frac{\partial u_i}{\partial x_j}-\tilde{\sigma}_{ij}\frac{\partial\bar{u}_i}{\partial x_j}$$

Conversion of SFS energy to internal energy by viscous dissipation

SFS viscous stress term

$$a_5=\frac{\partial(\tilde{\sigma}_{ij}\bar{u}_i-\bar{\sigma}_{ij}\bar{u}_i)}{\partial x_j}$$

$$a_6=\frac{\partial(\tilde{q}_j-\bar{q}_j)}{\partial x_j}$$

SFS heat flux term

The typical assumptions $\tilde{\sigma}_{ij}\approx\bar{\sigma}_{ij}$ *and* $\tilde{q}_j\approx\bar{q}_j$ lets a_5 and a_6 to vanish.

Large Eddy Simulation

Modeling of unresolved Scales

Two Classes of unresolved Scales

1. Resolved sub-filter scales (SFS) are scales with wave numbers large than the cutoff number but their effects are damped by filters with non-local in wave space such as box or Gaussian filters. They have to be treated applying filter reconstruction.
2. Sub-grid scales(SGS) are scale which are smaller than the cutoff filter width and they have to be either modeled or, for implicit LES, the numerical effects are assumed to mimic the physics of the unresolved turbulent motions.

Generally, SGS models use the well-established eddy viscosity approach and the first was the Smagorinsky model which assumes energy production and dissipation of the small scale are in equilibrium:

$$\nu_t = \left(C_s \Delta_g\right)^2 \sqrt{2 \bar{S}_{ij} \bar{S}_{ij}} = \left(C_s \Delta_g\right)^2 |S|$$ with Δ_g the grid size and C_s a constant

However, this constant turned out to be application dependent.

Modeling of unresolved Scales

Two Classes of unresolved Scales

German developed a dynamic model which applies two filters, a grid filter, named $\bar{\cdot}$ and a test filter $\hat{\cdot}$.

The resolved stress tensor $\Im_{ij} = T_{ij}^r - \hat{\tau}_{ij}^r$ is called the Germano identity.

$T_{ij}^r = \widehat{u_i u_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j$ is the residual stress tensor for the test filter scale and $\hat{\tau}_{ij}^r = \widehat{u_i u_j} - \widehat{\bar{u}_i \bar{u}_j}$ the one for the grid filter.

The Germano identity $\Im_{ij}$ represents the contribution to the sub-grid scales stresses by length scales smaller than the test filter width $\hat{\Delta}_t$ but larger than the grid filter width $\bar{\Delta}_g$. However, this turned out to become numerically unstable and additional averaging of the error in the minimization is often applied to yield:

$$C_s^2 = \frac{\Im_{ij} 2\bar{\Delta}^2 \left\langle \overline{\left|\hat{\bar{S}}\right|\hat{\bar{S}}_{ij} - \left(\frac{\hat{\Delta}}{\bar{\Delta}}\right)^2 \left|\bar{\hat{S}}\right|\bar{\hat{S}}_{ij}} \right\rangle}{2\bar{\Delta}^2 \left\langle \overline{\left|\hat{\bar{S}}\right|\hat{\bar{S}}_{ij} - \left(\frac{\hat{\Delta}}{\bar{\Delta}}\right)^2 \left|\bar{\hat{S}}\right|\bar{\hat{S}}_{ij}} \right\rangle 2\bar{\Delta}^2 \left\langle \overline{\left|\hat{\bar{S}}\right|\hat{\bar{S}}_{ij} - \left(\frac{\hat{\Delta}}{\bar{\Delta}}\right)^2 \left|\bar{\hat{S}}\right|\bar{\hat{S}}_{ij}} \right\rangle}$$

But still it is assumed that C_s is invariant of the scale. Averaging can be spatial as the original Germano procedure or time following Lagranian fluid trajectories.

Large Eddy Simulation

Boundary Conditions

Generation/Choice of inlet boundary conditions greatly affects the accuracy of LES and is as such a complex problem. Methods may be categorized in two classes.

1. Synthetic generation of turbulence by Fourier techniques, principle orthogonal decomposition or vortex methods. Although these techniques aim at the reconstruction of the turbulent field at inlet and simply the specification of turbulence parameters such as turbulent kinetic energy or dissipation rate, their main drawback is that the don't satisfy the physics of a flow governed by Navier-Stokes equations.
2. Prediction of turbulent data base library applying cyclic domains, pre-prepared library and internal mapping. However, the pre-calculation of turbulent inflow turbulence requires large computational effort.

In any case, the better the inflow conditions, the more accurate the LES results.

Direct Numerical Simulation

Basic Features

Solution of Navier-Stokes equations without averaging or approximations, i.e. only discretization errors which can be estimated/controlled appear

- Domain size at least a couple of times larger than the distance *L* over which fluctuations are correlated (*L*= largest eddy scale)
- Resolution needs to capture kinetic energy dissipation entirely which requires grid sizes smaller than the Kolmogorov scale, η
- If homogenous isotropic turbulence assumption holds, a uniform grid is adequate which yields the number of grid points in each direction:

$$L/\eta \sim O\left(\mathrm{Re}_L^{3/4}\right); \ \mathrm{Re}_L = UL/\nu$$

- For transient 3 D calculations (time step scales with grid size for stability/accuracy reasons)

$$L/\eta \sim O\left(\mathrm{Re}_L^{3}\right)$$

- Obviously, CPU and RAM requirements limit problem size dramatically;

Direct Numerical Simulation

Numerics

Time-marching methods are commonly used

- Explicit 2nd to 4th order accurate (Runge-Kutta)
- Implicit 2nd order (Crank-Nicolson)

Scheme has to be conservative, including kinetic energy

Spatial discretization schemes should have low dissipation

- Schemes which apply upwind differencing tend to be overly diffusive and may cause errors larger than molecular diffusion!
- High-order finite difference
- Spectral methods (use Fourier series to estimate derivatives)
  - Mainly useful for simple (periodic) geometries (FFT)
  - Use spectral elements instead
- Galerkin (FE Methods)

Direct Numerical Simulation

Challenges

Storage for states at intermediate time steps, particularly in cases finite rate chemistry is required

Total discretization error and turbulence spectrum

- Total error: both order of discretization and values of derivatives (spectrum)

  → measure of total error: integrate over whole turbulent spectrum

Error estimation difficult due to (unstable) nature of turbulent flow

  → often statistical properties of two solutions are compared (turbulent spectrum)

Initial conditions and methods of their generation are extremely important; i.e. the are \`remembered' for a significant number of "eddy-turnover" times

Generation of boundary conditions ng boundary conditions

- Periodic ones for simple problems, convective of non-reflective ones for realistic cases

What you shouldn't forget

- General description of turbulence
- Difference between Reynolds-averaging and Favre-averaging
- Kolmogorov Hypothesis
- Difference between 1-equation and 2-equation turbulence models
- Difference between k-ε, and k-ω SST
- Difference between eddy-viscosity models and Reynolds-Stress models
- General idea about LES and the importance of filtering process and sub-grid models
- General idea about DNS